TESTING CHEBYSHEV'S BIAS FOR PRIME NUMBERS UP TO $5*10^{15}$

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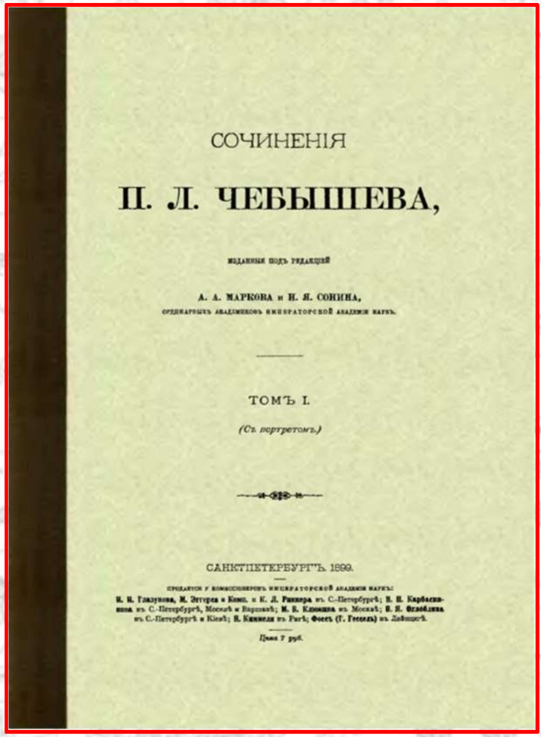
GOALS AND TARGETS FOR THE PROJECT

- *To test Chebyshev's Bias* ***for 15 "most biased prime number races"***
- ***To extend the range tested by mathematicians 5000 times to $5*10^{15}$*** ***(****$5*10^{15}$ – the upper bound and the last number of the tested range)*
- ***To define exactly the main characteristics of all sign-changing zones*** *(known, as well as newly found), including their beginning, end and number of terms*
- ***To check newly discovered zones against predictions***
- ***To test and confirm all previously known sign-changing zones for*** $\Delta_{q,a,b}(x)$ *up to 10^{12}*
- ***To make all primary data available*** *to a wide group of mathematicians working in number theory* ***field through OEIS publication and deposit in author's own repository***
- ***To define all data in a uniform way and with unified format***

The main goal of the project was to test Chebyshev's Bias for 15 selected moduli and pairs of residues for prime numbers up to $5*10^{15}$.

LETTER FROM CHEBYSHEV TO FUSS (1853)



СОЧИНЕНІЯ

П. Л. ЧЕБЫШЕВА,

изданныя подъ редакціей

А. А. МАРКОВА и Н. Я. СОНИНА,

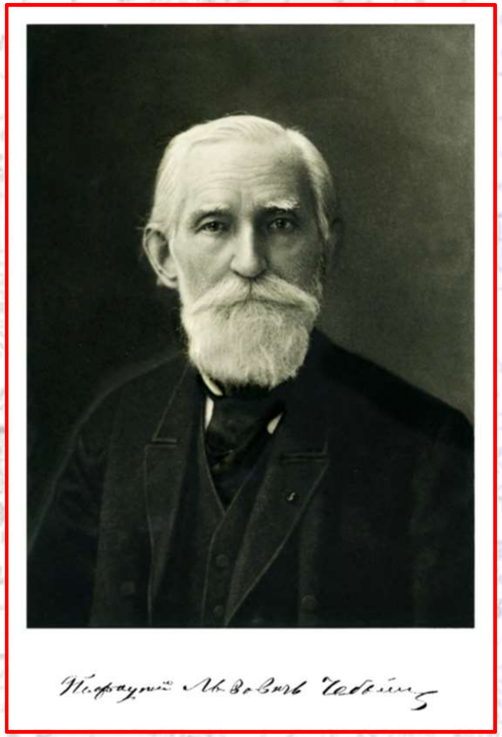
ординарныхъ академиковъ Императорской Академіи Наукъ.

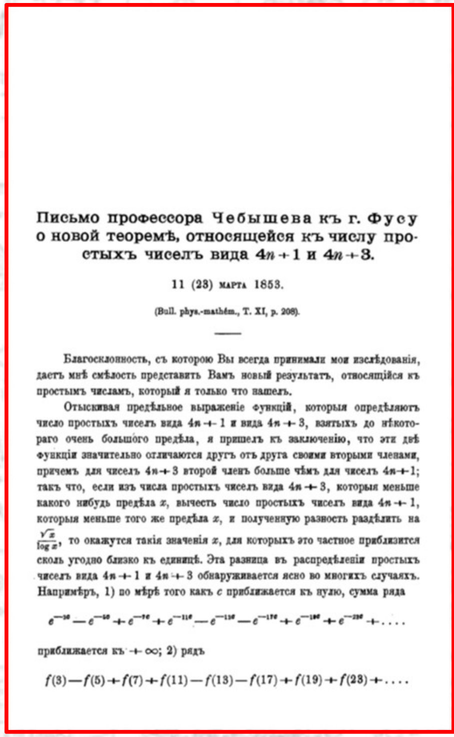
ТОМЪ I.

(Съ портретомъ.)

САНКТПЕТЕРБУРГЪ. 1899.

Цѣна 7 руб.





Письмо профессора Чебышева къ г. Фусу о новой теоремѣ, относящейся къ числу простыхъ чиселъ вида $4n+1$ и $4n+3$.

11 (23) марта 1853.

(Bull. phys.-mathém., T. XI, p. 208).

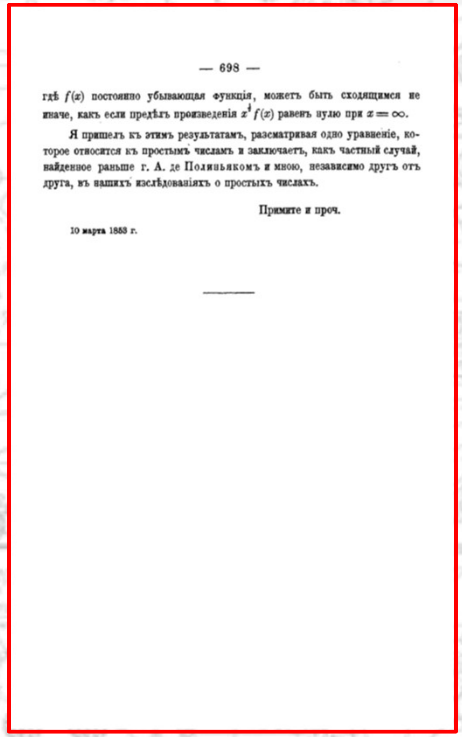
Благосклонность, съ которою Вы всегда принимали мои изслѣдованія, даетъ мнѣ смѣлость представить Вамъ новый результатъ, относящійся къ простымъ числамъ, который я только что нашелъ.

Отыскивая предѣльное выраженіе функцій, которыя опредѣляютъ число простыхъ чиселъ вида $4n+1$ и вида $4n+3$, взятыхъ до нѣкотораго очень большого предѣла, я пришелъ къ заключенію, что эти двѣ функціи значительно отличаются другъ отъ друга своими вторыми членами, причемъ для чиселъ $4n+3$ второй членъ больше чѣмъ для чиселъ $4n+1$; такъ что, если изъ числа простыхъ чиселъ вида $4n+3$, которыя меньше какого нибудь предѣла x, вычесть число простыхъ чиселъ вида $4n+1$, которыя меньше того же предѣла x, и полученную разность раздѣлить на $\frac{\sqrt{x}}{\log x}$, то окажутся такія значенія x, для которыхъ это частное приблизится сколь угодно близко къ единицѣ. Эта разница въ распредѣленіи простыхъ чиселъ вида $4n+1$ и $4n+3$ обнаруживается ясно во многихъ случаяхъ. Напримѣръ, 1) по мѣрѣ того какъ c приближается къ нулю, сумма ряда

$$e^{-3c}-e^{-5c}+e^{-7c}+e^{-11c}-e^{-13c}-e^{-17c}+e^{-19c}+e^{-23c}+\ldots$$

приближается къ $+\infty$; 2) рядъ

$$f(3)-f(5)+f(7)+f(11)-f(13)-f(17)+f(19)+f(23)+\ldots$$



— 698 —

гдѣ $f(x)$ постоянно убывающая функція, можетъ быть сходящимся не иначе, какъ если предѣлъ произведенія $x^{\frac{1}{2}} f(x)$ равенъ нулю при $x=\infty$.

Я пришелъ къ этимъ результатамъ, разсматривая одно уравненіе, которое относится къ простымъ числамъ и заключаетъ, какъ частный случай, найденное раньше г. А. де Полиньякомъ и мною, независимо другъ отъ друга, въ нашихъ изслѣдованіяхъ о простыхъ числахъ.

Примите и проч.

10 марта 1853 г.

Chebyshev's Bias (Chebyshev, 1853). ***"There is a notable difference in the splitting of the prime numbers between the two forms 4n + 3, 4n + 1: the first form contains a lot more than the second."***

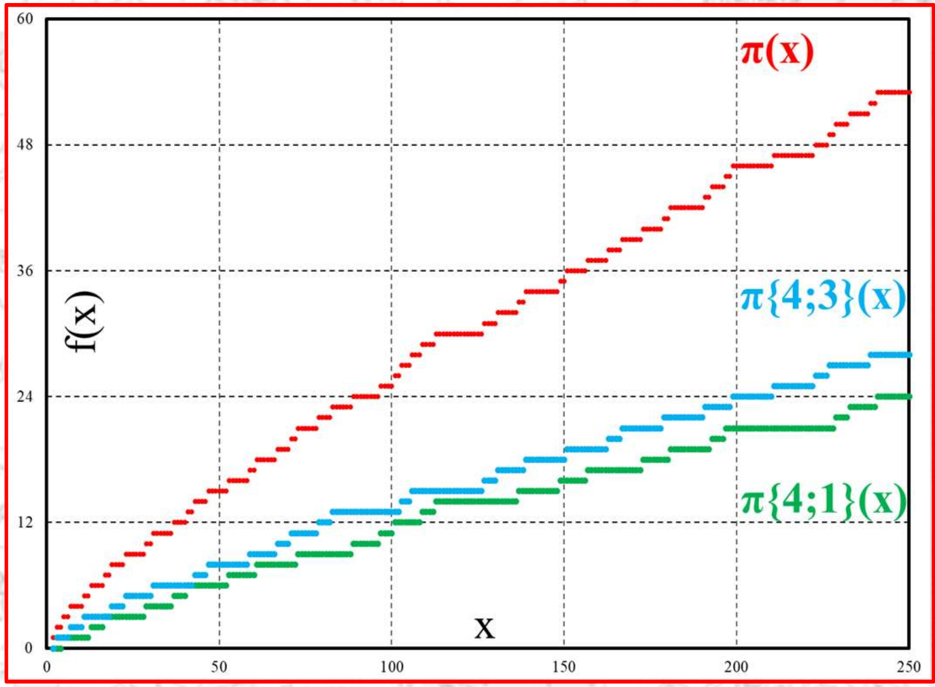
In 1853 Chebyshev suggested that there are always more primes of the form 4n + 3 than primes of the form 4n + 1.

** Pictures taken from: P. Chebyshev, "Works of P.L.Chebyshev" edited by A.A. Markov and N.Ya.Sonin, vol. 1, St.Petersburg, 1899, pp. Cover, Title & 697-698.*

CHEBYSHEV'S BIAS FOR TWO RESIDUES

$$\pi(x) = \pi_{4,3}(x) + \pi_{4,1}(x) + 1$$

$$\Delta_{4,3,1}(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$$



$\pi(x)$ – prime counting function

$q = 4$ – modulus

$a = 3$ and $b = 1$ – residues

$(a, q) = 1$, $(b, q)=1$

Chebyshev's Bias(1853): $\Delta_{4,3,1}(x) > 0$ for all x

- *Initial conjecture for $q = 4$, $a = 3$, $b = 1$*
- *Similar situation for $q = 3$, $a = 2$, $b = 1$*
- *«Prime number races»*

Chebyshev's Bias is easily formulated through prime counting function for two residues a and b modulo q.

CHEBYSHEV'S BIAS EXAMPLE FOR TWO RESIDUES MOD 4

x	π(x)	#{4n + 2}	#{4n + 3}	#{4n + 1}	Δ{4, 3, 1}	%
100	25	1	13	11	2	2.000%
200	46	1	24	21	3	1.500%
300	62	1	32	29	3	1.000%
400	78	1	40	37	3	0.750%
500	95	1	50	44	6	1.200%
600	109	1	57	51	6	1.000%
700	125	1	65	59	6	0.857%
800	139	1	71	67	4	0.500%
900	154	1	79	74	5	0.556%
1000	168	1	87	80	7	0.700%
2000	303	1	155	147	8	0.400%
3000	430	1	218	211	7	0.233%
4000	550	1	280	269	11	0.275%
5000	669	1	339	329	10	0.200%
6000	783	1	399	383	16	0.267%
7000	900	1	457	442	15	0.214%
8000	1007	1	507	499	8	0.100%
9000	1117	1	562	554	8	0.089%
10,000	1229	1	619	609	10	0.100%
20,000	2262	1	1136	1125	11	0.055%

- *The phenomena is* ***small, but permanent***
- *Effective percentage has* ***tendency to decrease***
- *At Chebyshev's times and 100 years after* ***no negative zones for*** $\Delta_{4,3,1}$ ***were known***
- *Only in 1957* ***the first and the second zones were discovered***

The first and second zones where Chebyshev's Bias was violated were discovered only in 1957 – more than 100 years after the letter to Fuss.

MAIN WORKS IN CHEBYSHEV'S BIAS AREA

1853 ***Letter from P.L. Chebyshev to P.N. Fuss***
1914 ***J. E. Littlewood, «Sur la distribution des nombres premiers»***
1957 *J. Leech, «Note on the distribution of prime numbers»*
1959 *D. Shanks «Quadratic Residues and the Distribution of Primes»*
1962 ***S. Knapowski and P. Turán, «Comparative Prime-Number Theory»***
1978 *C. Bays u R. Hudson, «Details of the first region of integers x with π{3,2}(x) < π{3,1}(x)»*
1978 *R. H. Hudson u C. Bays, «The appearance of tens of billion of integers x with π{24, 13}(x) < π{24, 1}(x) in the vicinity of 10^12»*
1979 *C. Bays u R. H. Hudson, «Numerical and graphical description of all axis crossing regions for the moduli 4 and 8 which occur before 10^12»*
1994 ***M. Rubinstein u P. Sarnak, «Chebyshev's Bias»***
2001 ***C. Bays, K. Ford, R. H. Hudson u M. Rubinstein, «Zeros of Dirichlet L-functions near the Real Axis and Chebyshev's Bias»***
2001 *K. Ford u R. H. Hudson, «Sign changes in pi{q;a}(x) - pi{q;b}(x)»*
2006 *A. Granville u G. Martin, «Prime Number Races»*
2012 *G. Martin u J. Scarfy, «Comparative Prime Number Theory»*
2013 ***D. Fiorilli u G. Martin, «Inequities in the Shanks-Renyi prime number race: an asymptotic formula for the densities»***

«Chebyshev's conjecture was the origin for a big branch of modern Number Theory, namely, comparative prime-number theory» as was written by S.V. Konyagin (Russia) and K. Ford (USA) in a joint paper.

CHEBYSHEV'S BIAS AND OTHER THEOREMS

Dirichlet prime number theorem for arithmetic progression *(Dirichlet, 1837). Let a, q $\in Z^+$ be such that gcd(a, q) = 1. Then there are infinitely many prime numbers p such that $p \equiv a \pmod q$. Therefore, as a result:*

$$\frac{\#\{\text{primes } qn + a \le x\}}{\#\{\text{primes } qn + b \le x\}} \to 1 \ (\text{as } x \to \infty)$$

Theorem (Littlewood, 1914). *There are arbitrarily large values of x for which there are more primes of the form 4n + 1 up to x than primes of the form 4n + 3. In fact, there are arbitrarily large values of x for which:*

$$\#\{primes\ 4n + 1 \le x\} - \#\{\text{primes } 4n + 3 \le x\} \ge \frac{1}{2}\frac{\sqrt{x}}{\ln x}\ \ln\ln\ln x$$

Conjecture (Knapowski and Turán, 1962). *As $X \to \infty$, the percentage of integers $x \le X$ for which there are more primes of the form 4n + 3 up to x than of the form 4n + 1 goes to 100%.*

Theorem (Kaczorowski, Rubinstein-Sarnak, 1994). *If the Generalized Riemann Hypothesis GRH is true, then the Knapowski-Turán Conjecture is false.*

The connection between Chebyshev's Bias and Generalized Riemann Hypothesis (GRH) was proven in 1994.

CHEBYSHEV'S BIAS AND OTHER THEOREMS

Generalized Riemann Hypothesis (GRH) (Piltz, 1884): *For any χ mod q and all complex $s = \sigma + it$ such as $0 \le \sigma \le 1$ and $L(\sigma + it, \chi) = 0$, all the non-trivial zeroes of the Dirichlet L-function $L(s, \chi)$ ($Re(s) > 1$) lie on the straight line $Re(s) = 1/2$.*

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_{p\ primes}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$$

Dirichlet L-function for "race of primes 4n + 3 vs. primes 4n + 1" (Re(s) > 1):

$$L(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \cdots\ldots$$

Therefore, the sum over primes in arithmetic progression is equivalent to the sum over zeros of Dirichlet L-function.

Rubinstein and Sarnak (1994): *The sum over primes in arithmetic progressions results into:*

$$\sum_{k\ge 1}\sum_{\substack{k\le x \\ p^k\equiv a \bmod q}}\frac{1}{k} = \pi(x;q,a) + \frac{1}{2}\left|\{p \le \sqrt{x}: p^2 \equiv a \bmod q\}\right| + error$$

The second term in the formula is the source of Chebyshev's Bias.

Chebyshev's Bias (modern formulation): *There are more primes of the form **qn + a** than of the form **qn + b** if **a** is non-square and **b** is a square residue modulo **q**.*

The source and the origin of Chebyshev's Bias is the presence of the square residue b among residues modulo q.

DISPROVAL OF KNAPOWSKI-TURÁN CONJECTURE

Maximum percentage of values of $x \leq X$ for which $\pi_{4,1}(x) > \pi_{4,3}(x)$

Range	Max %		
$0\text{-}10^7$	2.6%	⬇	Leech: 1957
$10^7\text{-}10^8$	0.6%	⬇	Lehmer: 1969
$10^8\text{-}10^9$	0.1%	⬇	Lehmer: 1969
$10^9\text{-}10^{10}$	1.6%	⬆	Bays & Hudson: 1979
$10^{10}\text{-}10^{11}$	2.8%	⬆	Bays & Hudson: 1979-1996

- *With exact formulation of Knapowski-Turán conjecture in 1962* ***the extensive search for $\Delta_{q,a,b}$ sign-changing zones started*** *for various moduli and residues*
- ***It became clear that Knapowski-Turán conjecture was fa****lse after a number breakthrough works and papers of C. Bays and R.H. Hudson (USA) who discovered several new sign-changing zones for $\Delta_{4,3,1}$ between 1979 and 1996*

Empirical data supported Knapowski-Turán conjecture up to 10^9 only. After Bays and Hudson research it became clear that it was wrong.

EMPIRICAL RESULTS: 1957-1996 (q = 3, 4 & 8)

Status of $\Delta_{q,a,b}(x)$ sign-changing zones search from 1957 to 1996

q	#	b	a	Beginning	Discovered
3	1	1	2	608,981,813,029	Bays & Hudson, 1978
4	1	1	3	26,861	Leech, 1957
4	2	1	3	616,841	Leech, 1957
4	3	1	3	12,306,137	Lehmer, 1969
4	4	1	3	951,784,481	Lehmer, 1969
4	5	1	3	6,309,280,709	Bays & Hudson, 1979
4	6	1	3	18,465,126,293	Bays & Hudson, 1979
4	7	1	3	1,488,478,427,089	Bays & Hudson, 1996
8	1	1	3	Not known up to 10^{12}	Not discovered
8	1	1	5	588,067,889	Bays & Hudson, 1979
8	2	1	5	35,615,130,497	Bays & Hudson, 1979
8	1	1	7	Not known up to 10^{12}	Not discovered

7 known zones

- *The search for new zones had been very slow - **sometimes decades** passed between the discoveries*
- ***Several outstanding mathematicians such as J. Leech** ("Leech lattice"), D.H. Lehmer ("Lucas-Lehmer primality test") and C. Bays & R.H. Hudson (prime number research and estimates for "Skewes number") **contributed greatly to the search***
- ***Most sign-changing zones (7) were found for $\Delta_{4,3,1}$***

There had been extensive search for Δ sign-changing zones up to 10^{12} from 1957 to 1996.

EMPIRICAL RESULTS: 1957-1996 (q = 12 & 24)

Status of $\Delta_{q,a,b}(x)$ sign-changing zones search from 1957 to 1996

q	#	b	a	Beginning	Discovered	
12	1	1	5	Not known up to 10^{12}	Not discovered	
12	1	1	7	Not known up to 10^{12}	Not discovered	
12	1	1	11	Not known up to 10^{12}	Not discovered	
24	1	1	5	Not known up to 10^{12}	Not discovered	
24	1	1	7	Not known up to 10^{12}	Not discovered	
24	1	1	11	Not known up to 10^{12}	Not discovered	
24	1	1	13	«Around 10^{12}»	Bays & Hudson, 1978	*Not defined exactly*
24	1	1	17	Not known up to 10^{12}	Not discovered	
24	1	1	19	Not known up to 10^{12}	Not discovered	
24	1	1	23	Not known up to 10^{12}	Not discovered	

- *Apart from $\Delta_{24,13,1}$* ***there had been no other found Δ sign-changing zones for q = 12 & 24***
- *For $\Delta_{24,13,1}$* ***the first zone was defined only approximately*** *without exact boundaries and number of terms*

Mod 12 and 24 presented a major problem as there had been almost nothing discovered and known about them.

EMPIRICAL RESULTS: 1996-2016

Status of Δq,a,b(x) sign-changing zones search from 1996 to 2016

q	#	b	a	Beginning	Discovered
3	2	1	2	6,148,171,711,663	Johnson, 2011
8	1	1	7	192,252,423,729,713	Martin, 2016

Found with mistakes

Only first point found

- ***New zones were discovered quite rarely***
- ***The range beyond 10^{12} was beyond the technical capabilities for a long time***
- *Both Johnson and Martin were programmers, not mathematicians*
- ***"Practice Is the Sole Criterion of Truth":*** *no theoretical model would ever disprove the numerically confirmed $\Delta_{q,a,b}$ sign-changing zones*
- *Direct numerical calculations for $\Delta_{q,a,b}$ sign-changing zones* ***have absolute accuracy***

In 20+ years since 1996 there have been only two sign-changing zones found, although with incomplete or inaccurate information.

LOGARITHMIC DENSITY/PROBABILITY OF $\pi_{4,3}(x) > \pi_{4,1}(x)$

Theorem (Rubinstein and Sarnak, 1994). *As $X \to \infty$,*

$$\frac{1}{\log X} \sum_{\substack{x \leq X \\ \pi_{4,3}(x) > \pi_{4,1}(x)}} \frac{1}{x} \to 0.9959 \ldots$$

In other words, Chebyshev was right 99.59% of the time!

Theorem (Rubinstein and Sarnak, 1994) *Let $(a;q) = (b;q) = 1$ such that $a \neq b \bmod q$. The logarithmic density*

$$\delta(q;a,b) := \lim_{X \to \infty} \frac{1}{\log X} \int_{\substack{t \in [2,X] \\ \pi(t;q,a) > \pi(t;q,b)}} \frac{dt}{t}$$

exists and is positive.

In 1994 the existence of positive logarithmic density, for Δ, meaning "the probability that $\pi_{q,a}(x) > \pi_{q,b}(x)$" was proved.

THE MOST "UNFAIR PRIME NUMBER RACES"

The "most unfair prime number races" (Fiorilli & Martin) & status (2013)

#	q	b	a	δ(q;a,1)	Status (2013)
1	24	1	5	0.999988	Not found up to 10^{12}
2	24	1	11	0.999983	Not found up to 10^{12}
3	12	1	11	0.999977	Not found up to 10^{12}
4	24	1	23	0.999889	Not found up to 10^{12}
5	24	1	7	0.999834	Not found up to 10^{12}
6	24	1	19	0.999719	Not found up to 10^{12}
7	8	1	3	0.999569	Not found up to 10^{12}
8	12	1	5	0.999206	Not found up to 10^{12}
9	24	1	17	0.999125	Not found up to 10^{12}
10	3	1	2	0.999063	Known up to 10^{12}
11	8	1	7	0.998939	Not found up to 10^{12}
12	24	1	13	0.998722	Known up to 10^{12}
13	12	1	7	0.998606	Not found up to 10^{12}
14	8	1	5	0.997395	Known up to 10^{12}
15	4	1	3	0.995928	Known up to 10^{12}

- ***Fundamental 2013 research*** *by Fiorilli and Martin on logarithmic densities*
- *Logarithmic densities were calculated and ranked* ***for 120 top "prime number races"***
- ***Top 15 were selected for test*** *within the scope of this project*

Not defined exactly

In 2013 the most "unfair prime number races" were theoretically defined, 15 of which were selected for this project up to 10^{15}.

PREDICTIONS OF NEW ZONES: q = 3, 4 & 8

Predictions of possible $\Delta_{q,a,b}(x)$ sign-changing zones up to 10^{20}

q	#	b	a	Beginning	Made by	
q=3	2	1	2	$6.15*10^{12}$	Bays & Hudson, 2001	CHECK!
q=3	3	1	2	$3.97*10^{19}$	Bays & Hudson, 2001	
q=3	3	1	2	$3.97*10^{19}$	Ford & Hudson, 2001	
q=4	8	1	3	$9.32*10^{12}$	Bays & Hudson, 2001	CHECK!
q=4	9	1	3	$9.97*10^{17}$	Deléglise, Dusart & Roblot, 2004	
q=8	1	1	3	$6.82*10^{18}$	Ford & Hudson, 2001	
q=8	1	1	5	$1.93*10^{14}$	Ford & Hudson, 2001	CHECK!
q=8	2	1	5	$9.32*10^{14}$	Ford & Hudson, 2001	CHECK!
q=8	1	1	7	$1.93*10^{14}$	Bays & Hudson, 2001	CHECK!
q=8	1	1	7	$1.93*10^{14}$	Ford & Hudson, 2001	CHECK!

- *One of the main goals of the project was* ***to check the predictions for new sign-changing zones made in the beginning of 2000s***
- *Some predictions (>10^{18})* ***were located far beyond the technical capabilities of that time***
- *Even today working above 10^{18} requires* ***the use of supercomputers with many cores and efficient multi-threading***

For q = 3, 4 and 8 the existence of six Δ sign-changing zones were predicted up to the 10^{15} – the upper boundary of the project.

PREDICTIONS OF NEW ZONES: q = 12 & 24

Predictions of possible $\Delta_{q,a,b}(x)$ sign-changing zones up to 10^{20}

q	#	b	a	Beginning	Predicted by
q=12	1	1	5	$9.84*10^{16}$	Ford & Hudson, 2001
q=12	1	1	7	$9.78*10^{16}$	Ford & Hudson, 2001
q=12	1	1	11	None < 10^{20}	Ford & Hudson, 2001
q=24	1	1	5	None < 10^{20}	Ford & Hudson, 2001
q=24	1	1	7	None < 10^{20}	Ford & Hudson, 2001
q=24	1	1	11	None < 10^{20}	Ford & Hudson, 2001
q=24	1	1	13	$6.74*10^{14}$	Ford & Hudson, 2001
q=24	1	1	17	$6.18*10^{14}$	Ford & Hudson, 2001
q=24	2	1	17	$7.11*10^{14}$	Ford & Hudson, 2001
q=24	1	1	19	$7.15*10^{14}$	Ford & Hudson, 2001
q=24	1	1	23	$7.44*10^{18}$	Ford & Hudson, 2001

CHECK!

CHECK!

CHECK!

CHECK!

- *One of the main goals of the project was **to check the predictions for new sign-changing zones made in the beginning of 2000s***
- *The situation with q = 12 and 24 was similar: some predictions ($>10^{18}$) **were located far beyond the technical capabilities of that time***

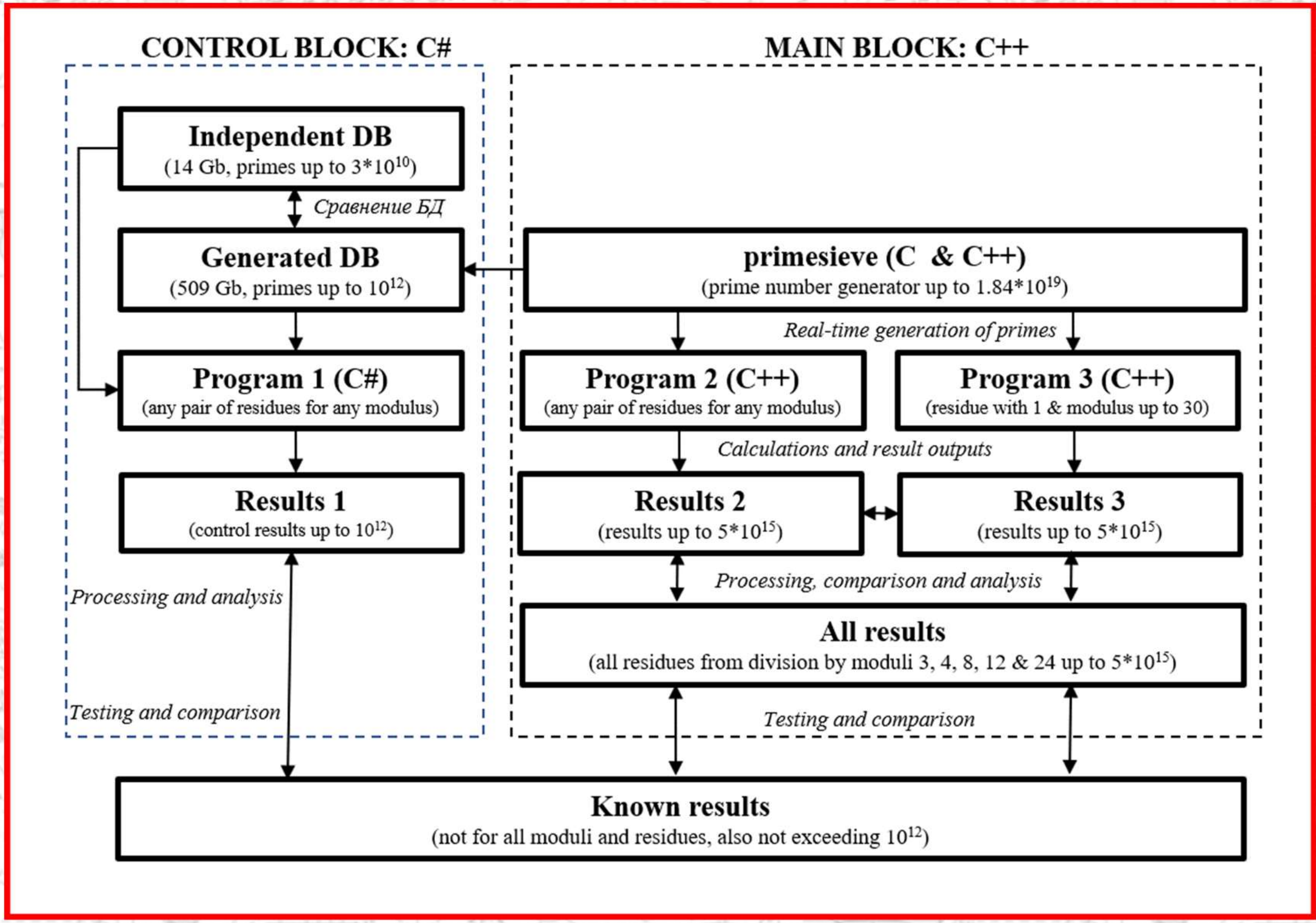
For q = 12 and 24 the existence of four Δ sign-changing zones were predicted up to the 10^{15} – the upper boundary of the project.

TECHNICAL DIFFICULTIES

- ***Ranges above 10^{15} seemed incredibly high 17 years ago*** *(in 2001) when Bays & Hudson summarized their 25-year effort in Chebyshev's Bias area*
- ***The direct brute-force method was extremely resource-consuming as well as sensitive to non-stop execution***
- ***Fast and reliable prime number generators*** *that were capable of working with large primes above 10^{12} and generate them without omissions and mistakes* ***were absent***
- *The alternative way of getting primes –* ***the preliminary generation with further database storage, required enormous memory size*** *(hundreds of terabytes or even petabytes) and barely allowed to move above 10^{12} leaving alone 10^{15} up*
- ***Fast and affordable servers*** *capable to work without mistakes and non-stop* ***24 x*** *7 for many weeks and months* ***were required***
- ***Many predicted points were located around 10^{18} – far above 10^{15},*** *that also reduced substantially the desire for implementation*
- *To work above 10^{18}* ***fast supercomputers with many cores and efficient multi-threading were required***

The direct brute force method to test Chebyshev's Bias even up to 10^{15} was difficult till recent advances in software and hardware development.

PROJECT TECHNICAL SET-UP



- ***2 main C++ programs***
- ***Primes up to $1.8*10^{19}$*** *(2^{64}) could be tested*
- ***Control C# program*** *with 10^{12} database to check*
- ***4 consecutive ranges to test:*** *10^{13}, 10^{14}, 10^{15}, $5*10^{15}$*
- ***At least 2 passes*** *for each range and "prime number race"*
- ***Project was extended to 10^{16} in May of 2018***
- *$5*10^{15}$ was reached in August 2018*
- *Data double-checked till November 2018*

Several C++ & C# programs were written for the project. The fastest known prime number generator "primesieve" was used for tests.

RESULTS: q = 3 (primes and values of n for primes)

Sign-changing zones for q = 3: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS	
q = 3	1	1	2	608,981,813,029	610,968,213,787	20,590	A297006	
q = 3	2	1	2	6,148,171,711,663	6,156,051,951,677	63,733	A297006	NEW! Ⓥ *6.15*10^12*
Total	**2**	**1**	**2**			**84,323**	**A297006**	

Sign-changing zones for q = 3: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS	
q = 3	1	1	2	23,338,590,792	23,411,791,034	20,590	A297005	
q = 3	2	1	2	216,415,270,060	216,682,882,512	63,733	A297005	NEW! Ⓥ
Total	**2**	**1**	**2**			**84,323**	**A297005**	

- ***Second zone matched exactly*** *with that predicted by Bays & Hudson (2001) at $6.15*10^{12}$*
- *New A297006 and A297005 sequences* ***were registered with OEIS***

For q = 3 the 2nd Δ sign-changing zone was found that almost exactly matched a zone predicted back in 2001.

RESULTS: q = 4 (primes)

Sign-changing zones for q = 4: primes

q	№	b	a	Beginning	End	# Δ= -1	OEIS		
q = 4	1	1	3	26,861	26,861	1	A051025		
q = 4	2	1	3	616,841	633,797	90	A051025		
q = 4	3	1	3	12,306,137	12,382,313	150	A051025		
q = 4	4	1	3	951,784,481	952,223,473	396	A051025		
q = 4	5	1	3	6,309,280,709	6,403,150,189	6,205	A051025		
q = 4	6	1	3	18,465,126,293	19,033,524,533	6,524	A051025		
q = 4	7	1	3	1,488,478,427,089	1,494,617,929,603	14,189	A051025		
q = 4	8	1	3	9,103,362,505,801	9,543,313,015,309	391,378	A051025	NEW! V	$9.32*10^{12}$
q = 4	9	1	3	64,083,080,712,569	64,084,318,523,021	13,370	A051025	NEW! !	$9.97*10^{17}$
q = 4	10	1	3	715,725,135,905,981	732,156,384,107,921	481,194	A051025	NEW! !	
Total	**10**	**1**	**3**			**913,497**	**A051025**		

- ***The 8th zone happened lower*** *than was predicted by Bays & Hudson (2001) at $9.32*10^{12}$*
- ***The 9th & 10th zones were not expected up to 10^{18}***
- ***OEIS sequence*** *A051025 with only 30 terms was complemented and now includes 913,497 terms*

For q = 4 three new zones (8th, 9th & 10th) were discovered. According to the theoretical models the last two had not been expected below 10^{18}.

RESULTS: q = 4 (values of n for primes)

Sign-changing zones for q = 4: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS	
q = 4	1	1	3	2,946	2,946	1	A051024	
q = 4	2	1	3	50,378	51,622	90	A051024	
q = 4	3	1	3	806,808	811,528	150	A051024	
q = 4	4	1	3	48,517,584	48,538,970	396	A051024	
q = 4	5	1	3	293,267,470	297,424,714	6,205	A051024	
q = 4	6	1	3	817,388,828	841,415,718	6,524	A051024	
q = 4	7	1	3	55,152,203,450	55,371,233,730	14,189	A051024	
q = 4	8	1	3	316,064,952,540	330,797,040,308	391,378	A051024	NEW! V
q = 4	9	1	3	2,083,576,475,506	2,083,615,410,040	13,370	A051024	NEW! !
q = 4	10	1	3	21,576,098,946,648	22,056,324,317,296	481,194	A051024	NEW! !
Total	**10**	**1**	**3**			**913,497**	**A051024**	

- ***The 8th zone happened lower*** *than was predicted by Bays & Hudson (2001)*
- ***The 9th and 10th zone were not expected so low***
- ***OEIS sequence*** *A051024 with only 33 terms was complemented and now includes 913,497 terms*

For q = 4 three new zones (8th, 9th & 10th) were discovered. According to the theoretical models the last two had not been expected below 10^{18}.

RESULTS: q = 8 (primes)

Sign-changing zones for q = 8: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS	
q = 8	**1**	**1**	**3**	**Not found up to 5*10^{15}**				
q = 8	1	1	5	588,067,889	593,871,533	488	A297448	
q = 8	2	1	5	35,615,130,497	37,335,021,821	22,305	A297448	
q = 8	3	1	5	5,267,226,902,633	5,312,932,515,721	109,831	A297448	NEW! !
q = 8	4	1	5	5,758,938,230,761	5,768,749,719,461	48,229	A297448	NEW! !
q = 8	5	1	5	6,200,509,945,537	6,209,511,651,289	18,048	A297448	NEW! !
q = 8	6	1	5	192,189,726,613,273	194,318,969,449,909	465,274	A297448	NEW! V *1.93*10^{14}*
q = 8	7	1	5	930,525,161,507,057	932,080,335,660,277	186,057	A297448	NEW! V *9.32*10^{14}*
Total	**7**	**1**	**5**			**850,232**	**A297448**	
q = 8	1	1	7	192,252,423,729,713	192,876,135,747,311	234,937	A295354	NEW! V *1.93*10^{14}*
Total	**1**	**1**	**7**			**234,937**	**A295354**	

- ***Not a single zone discovered for*** $\Delta_{8,3,1}(x)$
- *Out of 5 discovered zones for* $\Delta_{8,5,1}(x)$ ***only the 6th and 7th (2 widest ones) were predicted correctly at*** *1.93*10^{14} and 9.32*10^{14} respectively*
- *The 1st zone for* $\Delta_{8,7,1}(x)$ ***was also predicted correctly at*** *1.93*10^{14}*
- ***4 new sequences were registered****: A297448, A297447, A295354 and A295353*

Out of 5 new discovered zones for $\Delta_{8,5,1}(x)$ theoretical models correctly predicted only 2. The prediction for $\Delta_{8,7,1}(x)$ was also confirmed.

RESULTS: q = 8 (values of n for primes)

Sign-changing zones for q = 8: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 8	**1**	**1**	**3**	**Not found up to $5*10^{15}$**					
q = 8	1	1	5	30,733,704	31,021,248	488	A297447		
q = 8	2	1	5	1,531,917,197	1,602,638,725	22,305	A297447		
q = 8	3	1	5	186,422,420,112	187,982,502,637	109,831	A297447	NEW!	!
q = 8	4	1	5	203,182,722,672	203,516,651,165	48,229	A297447	NEW!	!
q = 8	5	1	5	218,192,372,353	218,497,974,121	18,048	A297447	NEW!	!
q = 8	6	1	5	6,033,099,205,868	6,097,827,689,926	465,274	A297447	NEW!	V
q = 8	7	1	5	27,830,993,289,634	27,876,113,171,315	186,057	A297447	NEW!	V
Total	**7**	**1**	**5**			**850,232**	**A297447**		
q = 8	1	1	7	6,035,005,477,560	6,053,968,231,350	234,937	A295353	NEW!	V
Total	**1**	**1**	**7**			**234,937**	**A295353**		

- ***Not a single zone discovered for*** $\Delta_{8,3,1}(x)$
- *Out of 5 discovered zones for* $\Delta_{8,5,1}(x)$ ***only the 6th and 7th (2 widest ones) were predicted correctly***
- *The 1st zone for* $\Delta_{8,7,1}(x)$ ***was also predicted correctly***
- ***4 new sequences were registered****: A297448, A297447, A295354 and A295353*

Out of 5 new discovered zones for $\Delta_{8,5,1}(x)$ theoretical models correctly predicted only 2. The prediction for $\Delta_{8,7,1}(x)$ was also confirmed.

RESULTS: q = 12 (primes and values of n for primes)

Sign-changing zones for q = 12: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 12	1	1	5	25,726,067,172,577	25,727,487,045,613	8,399	A297355	NEW!	! $9.84*10^{16}$
Total	**1**	**1**	**5**			**8,399**	**A297355**		
q = 12	1	1	7	27,489,101,529,529	27,555,497,263,753	55,596	A297357	NEW!	! $9.78*10^{16}$
Total	**1**	**1**	**7**			**55,596**	**A297357**		
q = 12	**1**	**1**	**11**	**Not found up to $5*10^{15}$**					

Sign-changing zones for q = 12: values of n for primes (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS		
q = 12	1	1	5	862,062,606,318	862,108,594,325	8,399	A297354	NEW!	!
Total	**1**	**1**	**5**			**8,399**	**A297354**		
q = 12	1	1	7	919,096,512,484	921,242,027,614	55,596	A297356	NEW!	!
Total	**1**	**1**	**7**			**55,596**	**A297356**		
q = 12	**1**	**1**	**11**	**Not found up to $5*10^{15}$**					

- ***Not a single zone discovered for $\Delta_{12,1,1}(x)$***
- *Discovered zone for $\Delta_{12,5,1}(x)$* ***happened to be narrow and lower than predicted at*** *$9.84*10^{16}$*
- *Discovered zone for $\Delta_{12,7,1}(x)$* ***happened to be narrow and lower than predicted at*** *$9.78*10^{16}$*
- ***Four new OEIS sequences were registered*** *A297355, A297354, A297357 and A297356*

In $5*10^{15}$ range theoretical models failed to predict both discovered zones unknown before. This requires explanation and change in the models!

RESULTS: q = 24 (primes)

Sign-changing zones for q = 24: primes

q	№	b	a	Beginning	End	# Δ = -1	OEIS			
q = 24	**1**	**1**	**5**	**Not found up to 5*10^{15}**						
q = 24	**1**	**1**	**7**	**Not found up to 5*10^{15}**						
q = 24	**1**	**1**	**11**	**Not found up to 5*10^{15}**						
q = 24	1	1	13	978,412,359,121	989,462,029,561	9,920	A295356			
q = 24	2	1	13	1,005,578,970,337	1,009,517,096,641	22,648	A295356	NEW!	!	
q = 24	3	1	13	1,025,403,695,233	1,096,157,101,033	111,408	A295356	NEW!	!	
q = 24	4	1	13	648,452,989,927,609	649,632,972,248,893	202,195	A295356	NEW!	!	
q = 24	5	1	13	655,404,854,710,621	662,189,414,787,361	594,414	A295356	NEW!	V	*6.74*10^{14}*
q = 24	6	1	13	687,936,222,802,693	699,914,738,212,849	1,441,319	A295356	NEW!	V	
Total	**6**	**1**	**13**			**2,381,904**	**A295356**			
q = 24	1	1	17	617,139,273,158,713	618,051,990,355,993	73,201	A297450	NEW!	V	*6.18*10^{14}*
q = 24	2	1	17	709,763,768,223,841	714,186,411,923,009	773,982	A297450	NEW!	V	*7.11*10^{14}*
q = 24	3	1	17	772,451,788,864,537	772,739,867,710,897	116,739	A297450	NEW!	!	
Total	**3**	**1**	**17**			**963,922**	**A297450**			
q = 24	1	1	19	706,866,045,116,113	709,591,447,226,587	260,586	A298821	NEW!	!	
q = 24	2	1	19	716,328,072,795,619	725,993,117,452,657	833,790	A298821	NEW!	V	*7.15*10^{14}*
q = 24	3	1	19	731,496,205,367,611	733,085,386,984,849	306,557	A298821	NEW!	!	
q = 24	4	1	19	739,965,838,936,153	756,906,118,578,763	1,586,533	A298821	NEW!	!	
q = 24	5	1	19	761,403,326,459,539	766,164,822,666,883	449,524	A298821	NEW!	!	
Total	**5**	**1**	**19**			**3,436,990**	**A298821**			
q = 24	**1**	**1**	**23**	**Not found up to 5*10^{15}**						

- ***Six new OEIS sequences were registered*** *A295356, A295355, A297450, A297449, A298821 and A298820*

For q = 24 13 new zones were discovered for $\Delta_{24,13,1}(x)$, $\Delta_{24,17,1}(x)$ and $\Delta_{24,19,1}(x)$. None were found for $\Delta_{24,5,1}(x)$, $\Delta_{24,7,1}(x)$, $\Delta_{24,11,1}(x)$ & $\Delta_{24,23,1}(x)$.

RESULTS: q = 24 (values of n for primes)

Sign-changing zones for q = 24: values of n (π(x) function)

q	№	b	a	Beginning	End	# Δ = -1	OEIS	
q = 24	**1**	**1**	**5**	**Not found up to 5*10^{15}**				
q = 24	**1**	**1**	**7**	**Not found up to 5*10^{15}**				
q = 24	**1**	**1**	**11**	**Not found up to 5*10^{15}**				
q = 24	1	1	13	36,826,322,708	37,226,458,011	9,920	A295355	
q = 24	2	1	13	37,809,796,159	37,952,282,986	22,648	A295355	NEW! (!)
q = 24	3	1	13	38,526,874,563	41,082,097,577	111,408	A295355	NEW! (!)
q = 24	4	1	13	19,606,529,038,612	19,641,125,979,304	202,195	A295355	NEW! (!)
q = 24	5	1	13	19,810,330,673,460	20,009,166,153,467	594,414	A295355	NEW! (V)
q = 24	6	1	13	20,763,192,869,094	21,113,714,560,133	1,441,319	A295355	NEW! (V)
Total	**6**	**1**	**13**			**2,381,904**	**A295355**	
q = 24	1	1	17	18,687,728,175,380	18,714,528,041,257	73,201	A297449	NEW! (V)
q = 24	2	1	17	21,401,790,499,965	21,531,111,289,460	773,982	A297449	NEW! (V)
q = 24	3	1	17	23,232,693,876,716	23,241,097,440,243	116,739	A297449	NEW! (!)
Total	**3**	**1**	**17**			**963,922**	**A297449**	
q = 24	1	1	19	21,317,046,795,798	21,396,751,256,986	260,586	A298820	NEW! (!)
q = 24	2	1	19	21,593,726,305,432	21,876,231,682,201	833,790	A298820	NEW! (V)
q = 24	3	1	19	22,037,035,819,978	22,083,466,138,743	306,557	A298820	NEW! (!)
q = 24	4	1	19	22,284,455,265,595	22,779,076,769,443	1,586,533	A298820	NEW! (!)
q = 24	5	1	19	22,910,331,360,479	23,049,274,819,456	449,524	A298820	NEW! (!)
Total	**5**	**1**	**19**			**3,436,990**	**A298820**	
q = 24	**1**	**1**	**23**	**Not found up to 5*10^{15}**				

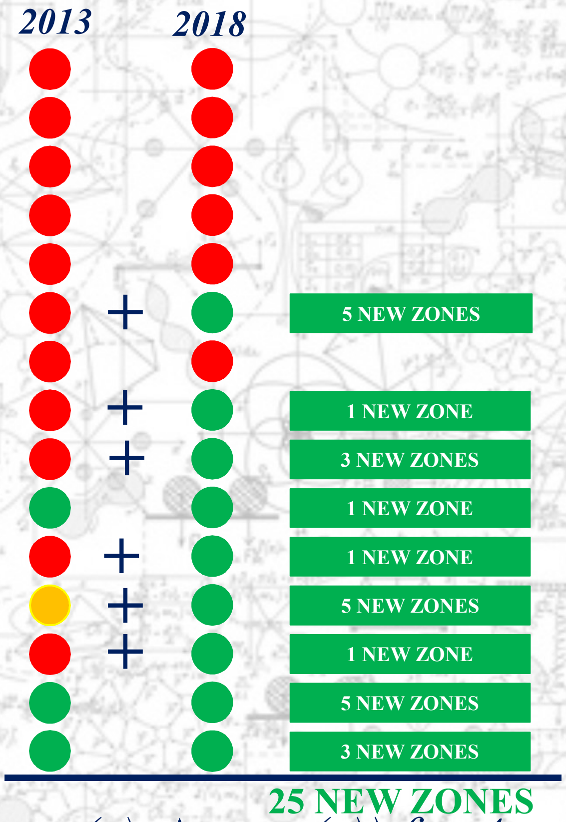
- ***6 sequences registered*** *A295356, A295355, A297450, A297449, A298821 & A298820*

For q = 24 13 new zones were discovered for $\Delta_{24,13,1}(x)$, $\Delta_{24,17,1}(x)$ and $\Delta_{24,19,1}(x)$. None were found for $\Delta_{24,5,1}(x)$, $\Delta_{24,7,1}(x)$, $\Delta_{24,11,1}(x)$ & $\Delta_{24,23,1}(x)$.

RESULTS

The most "unfair prime number races" – the largest δ(q;a,1) and status as of 2018

#	q	b	a	δ(q;a,1)	Status (2018)
1	24	1	5	0.999988	Not found up to $5*10^{15}$ (2018)
2	24	1	11	0.999983	Not found up to $5*10^{15}$ (2018)
3	12	1	11	0.999977	Not found up to $5*10^{15}$ (2018)
4	24	1	23	0.999889	Not found up to $5*10^{15}$ (2018)
5	24	1	7	0.999834	Not found up to $5*10^{15}$ (2018)
6	24	1	19	0.999719	Found up to $5*10^{15}$ (2018)
7	8	1	3	0.999569	Not found up to $5*10^{15}$ (2018)
8	12	1	5	0.999206	Found up to $5*10^{15}$ (2018)
9	24	1	17	0.999125	Found up to $5*10^{15}$ (2018)
10	3	1	2	0.999063	Known up to $5*10^{15}$ (2018)
11	8	1	7	0.998939	Found up to $5*10^{15}$ (2018)
12	24	1	13	0.998722	Known up to $5*10^{15}$ (2018)
13	12	1	7	0.998606	Found up to $5*10^{15}$ (2018)
14	8	1	5	0.997395	Known up to $5*10^{15}$ (2018)
15	4	1	3	0.995928	Known up to $5*10^{15}$ (2018)

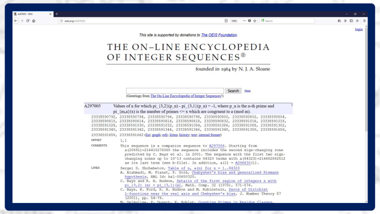


- ***Discovered 4 first ever zones*** *($\Delta_{12,5,1}(x)$, $\Delta_{12,7,1}(x)$, $\Delta_{24,17,1}(x)$, $\Delta_{24,19,1}(x)$) for 4 out of 15 most interesting and "unfair prime number races"*
- ***In total 25 new*** *$\Delta_{q,a,b}(x)$ sign-changing zones discovered*
- ***In total 18 sequences were registered or substantially extended with OEIS***
- ***Sign-changing zones for only 6 "most unfair prime number races" remain unknown***

Project implementation allowed to advance substantially in search for Δ sign-changing zones for the most interested "prime number races".

RESULTS: PUBLISHED DATA

A297005: $\pi(x)\{\Delta_{3,2,1}(x)=-1\}$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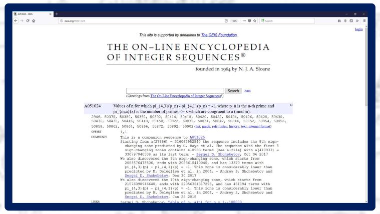
A051024: $\pi(x)\{\Delta_{4,3,1}(x)=-1\}$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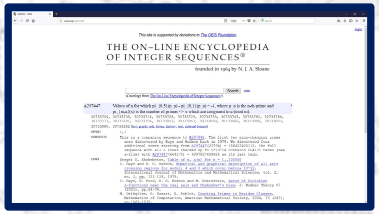
A297447: $\pi(x)\{\Delta_{8,5,1}(x)=-1\}$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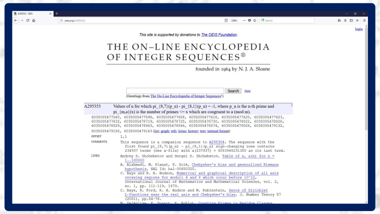
A295353: $\pi(x)\{\Delta_{8,7,1}(x)=-1\}$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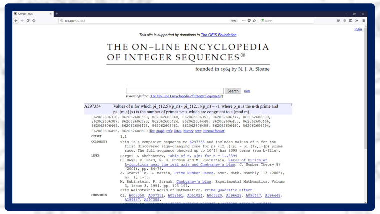
A297354: $\pi(x)\{\Delta_{12,5,1}(x)=-1\}$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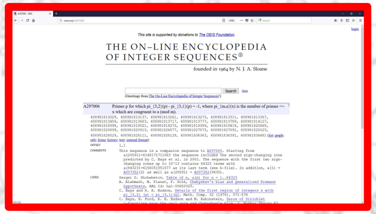
A297006: $p(x)\{\Delta_{3,2,1}(x)=-1\}$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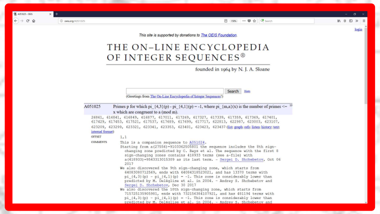
A051025: $p(x)\{\Delta_{4,3,1}(x)=-1\}$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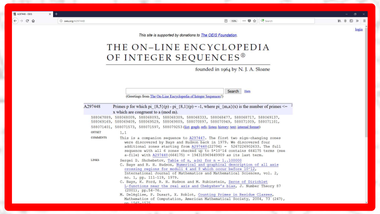
A297448: $p(x)\{\Delta_{8,5,1}(x)=-1\}$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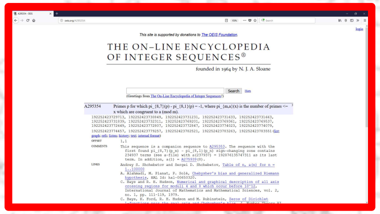
A295354: $p(x)\{\Delta_{8,7,1}(x)=-1\}$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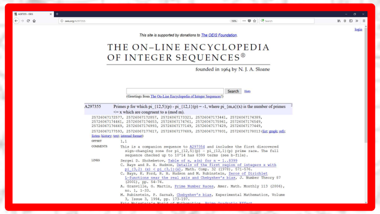
A297355: $p(x)\{\Delta_{12,5,1}(x)=-1\}$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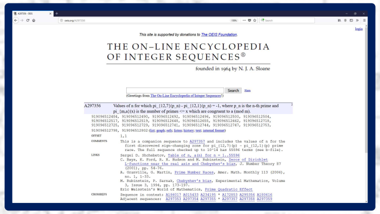
founded in 1964 by N. J. A. Sloane

A297356: $\pi(x)\{\Delta_{12,7,1}(x)=-1\}$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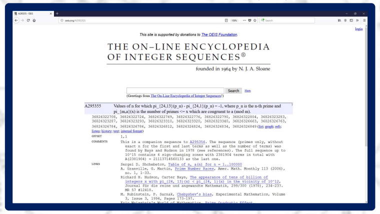
A295355: $\pi(x)\{\Delta_{24,13,1}(x)=-1\}$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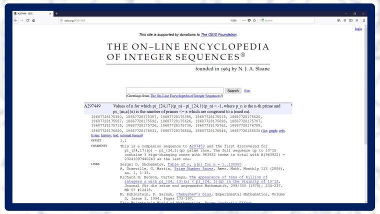
A297449: $\pi(x)\{\Delta_{24,17,1}(x)=-1\}$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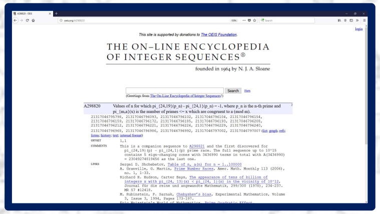
A298820: $\pi(x)\{\Delta_{24,19,1}(x)=-1\}$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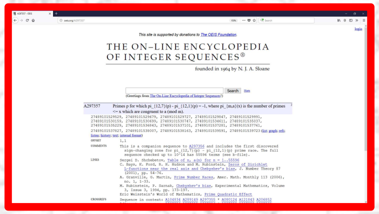
A297357: $p(x)\{\Delta_{12,7,1}(x)=-1\}$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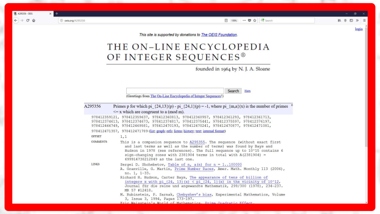
A295356: $p(x)\{\Delta_{24,13,1}(x)=-1\}$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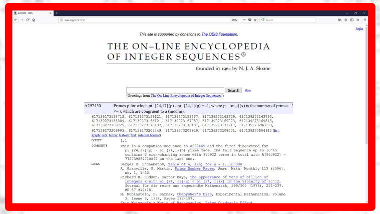
A297450: $p(x)\{\Delta_{24,17,1}(x)=-1\}$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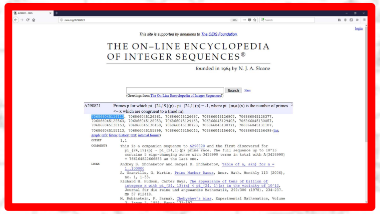
A298821: $\pi(x)\{\Delta_{24,19,1}(x)=-1\}$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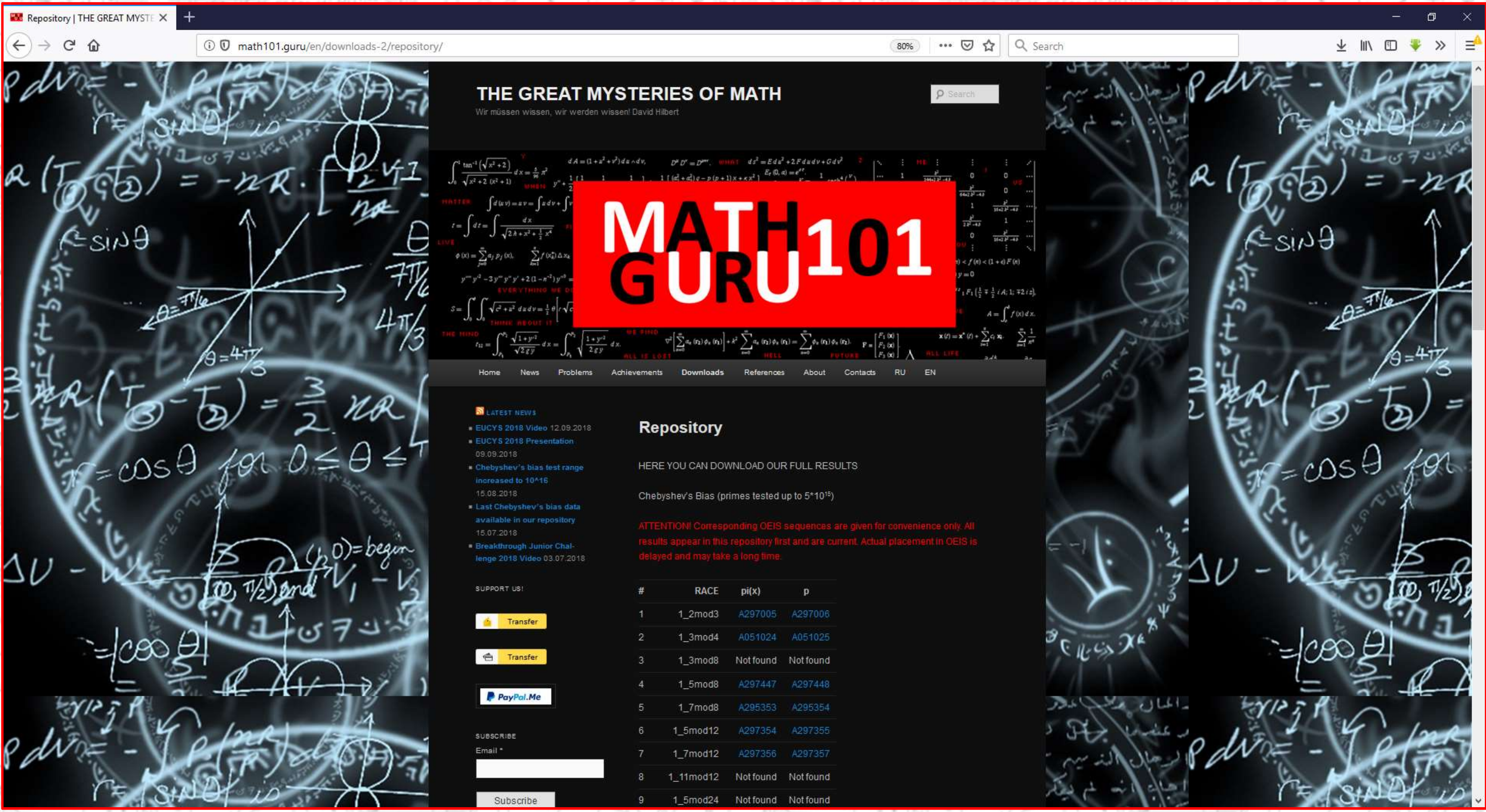
All data were published in The Online Encyclopedia of Integer Sequences (OEIS) as 18 separate sequences.

RESULTS: PUBLISHED DATA



All results are available at project repository at www.math101.guru (http://math101.guru/en/downloads-2/repository/).

RESULTS: CONCLUSIONS

- ***Chebyshev's Bias was tested up to $5*10^{15}$*** *for selected 15 "most biased prime number races", established theoretically in 2013*
- ***First sign-changing zones were discovered*** *for 4 "most biased prime number races" out of selected 15* ***(6 still remain unknown)***
- ***In total, 25 new sign-changing zones for delta were found***
- ***It was confirmed that theoretical models fail to predict small and narrow zones*** *that occur more frequently than assumed*
- *It was confirmed that* ***theoretical models predict big and wide zones relatively well***
- ***18 sequences were registered or substantially extended*** *with OEIS*
- ***All zones were accurately and exactly defined*** *(beginning, end, number of terms)*
- ***Full and complete data are available*** *to everybody*
- ***Created software allows to test Chebyshev's Bias up to 2^{64} ($1.8*10^{19}$)***
- ***The article is under work for submission to*** *«Mathematics of Computation»*

Project implementation allowed to extend substantially our knowledge on Chebyshev's Bias and define the accuracy of theoretical models.